Complex dielectric constant and classical harmonic oscillator

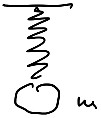
Now we will explore how light interacts with media.

Let's first talk about simple case - isotropic medium.

This isotropic medium has the following properties:

1) Medium contains „+" and „-" charges
2) Light wave or electromagnetic wave can act on this charges
3) Due to the presence of light or e/m wave the particles are oscillating.

Here we need to answer the question, why light of different colours interacts with medium differently. What are the principles of this interactions?



The interaction with light is going in 3 steps:

1) Local response of medium.
   That means each local oscillator starts moving under the external e/m field.
2) Re-emission by secondary waves.
   Since charges start to move, they become the sources of secondary e/m wave.
3) Superposition of secondary waves.
   We can imagine here, that original e/m wave is substituted by sum of secondary waves.

Let's discuss now how plane waves are propagating in dielectrics.

Maxwell's equations we already know:

$$\begin{cases} \bar{\nabla} \bar{D} = 0 \\ \bar{\nabla} \bar{B} = 0 \\ \left[\bar{\nabla} \bar{E}\right] = -\dfrac{\partial \bar{B}}{\partial t} \\ \left[\bar{\nabla} \bar{H}\right] = \dfrac{\partial \bar{D}}{\partial t} \end{cases}$$

Let's add two more equations, so-called material equations:

$$\bar{D} = \varepsilon_0 \varepsilon \bar{E} \quad (1)$$

ε - relative dielectric susceptibility, also known as dielectric constant.

$$\bar{B} = \mu_0 \bar{H} \quad (\mu = 1)$$

It is important to notice, we are considering media, where $\mu = 1$. That means we are going to discuss paramagnetic and diamagnetic media, but not ferromagnetic ($\mu > 1$)

Let's take plane waves:

$$\begin{cases} \bar{E} = \bar{E}_0 e^{i(\omega t - \bar{k}\bar{r})} \\ \bar{B} = \bar{B}_0 e^{i(\omega t - \bar{k}\bar{r})} \end{cases}$$

We are talking about monochromatic waves, i.e. $\omega = const$.

Little reminder

$$\bar{\nabla} \Rightarrow -i\bar{k}$$

$$\frac{\partial}{\partial t} \Rightarrow i\omega$$

In this case, Maxwell's equations can be written as following:

$$-i\bar{k}\bar{D} = 0 \qquad \bar{k}\bar{E} = 0$$ (through first material equation)

Other Maxwell's equations:

$-i[\bar{k}\,\bar{E}] = -i\omega\bar{B}$ $\qquad$ $-i[\bar{k}\,\bar{H}] = i\omega\bar{D}$

$(\bar{B}=\mu_0\bar{H})$

We can simplify:

$[\bar{k}\,\bar{E}] = \omega\bar{B}$ $\qquad$ $-[\bar{k}\,\bar{B}] = \underbrace{\mu_0\varepsilon_0}_{1/c^2}\varepsilon\omega\bar{E} = \frac{\omega}{c^2}\varepsilon\bar{E}$

We will multiply by $\bar{k}$

$$[\bar{k}[\bar{k}\bar{E}]] = \omega[\bar{k}\bar{B}]$$

$$\bar{k}(\underset{\overset{\shortparallel}{0}}{\bar{k}\bar{E}}) - k^2\bar{E} = -\frac{\omega^2}{c^2}\varepsilon\bar{E}$$

Hence we will have:

$$\boxed{\left(k^2 - \frac{\omega^2}{c^2}\varepsilon\right)\bar{E}=0}$$

The only non-trivial solution is when

$k^2 - \frac{\omega^2}{c^2}\varepsilon = 0$ $\qquad$ $\bar{E}=0$ is not interesting for us.

$$\Downarrow$$

$$\boxed{k=\frac{\omega}{c}\sqrt{\varepsilon}}$$

1) If we have transparent dielectric, then $\varepsilon>0$ is real number.

In this case:

$\sqrt{\varepsilon}=n \rightarrow$ we will call it refractive index

In fact

$\sqrt{\varepsilon}=n(\omega) \Rightarrow k=\frac{\omega}{c}n=\frac{\omega}{v}$

To remind, in vacuum $k=\frac{\omega}{c}$

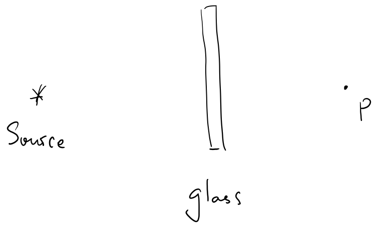
$n=\frac{c}{v}$ — in vacuum
— in medium

2) If we have absorbing dielectric, then ε is the complex number

$\sqrt{\varepsilon}=n'(\omega) \rightarrow$ complex refractive index

Feynman approach (On origin of refractive index)

$E_p = E_s + \Sigma E_i$

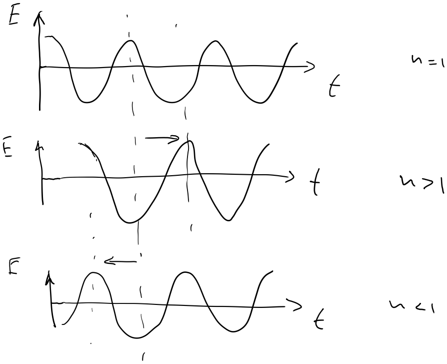


E_i - field of each charge

$$E_s = E_0 \cos\omega\left(t - \frac{z}{c}\right) = E_0 e^{i\omega\left(t - \frac{z}{c}\right)}$$

$$E_p = E_0 e^{i\omega\left[t - (n-1)\frac{\Delta z}{c} - \frac{z}{c}\right]} =$$

$$= \underbrace{e^{i\omega(n-1)\frac{\Delta z}{c}}}_{\text{phase / delay}} E_0 e^{i\omega\left(t - \frac{z}{c}\right)}$$



Instead of finding $E_p = E_s + \Sigma E_i$ we found the effect of plate is multiplying field by factor which shifts its phase. It is alright as we get same result by adding complex number.

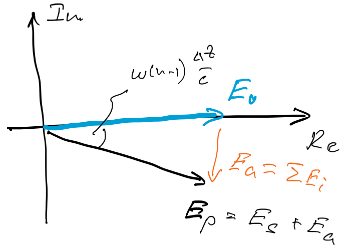
If $\Delta z \to 0$ $\quad e^x = 1 + x, \; x \to 0$

$$e^{-i\omega(n-1)\frac{\Delta z}{c}} \approx 1 - i\omega(n-1)\frac{\Delta z}{c}$$

$$\Downarrow$$

$$E_p = \underbrace{E_0 e^{i\omega\left(t - \frac{z}{c}\right)}}_{} - \underbrace{\frac{i\omega(n-1)\Delta z}{c} E_0 e^{i\omega\left(t - \frac{z}{c}\right)}}_{}$$

E_s ΣE_i



What is it disperssion?

$n(\omega)$ or $n'(\omega)$

Disperssion is frequency dependence of refractive index.

In order to understand why it is frequency dependent we need to explore electronic theory of disperssion.

Let's make an model of light propagation:

1) External e/m wave acts on neutral atom.

2) Atom becomes oscillating dipole.

3) This oscillating dipole emits e/m wave.

What would be the model of medium in which such e/m wave will propagate?

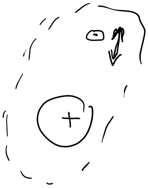
- isotropic
- homogeneous
- low density
- linear

We will discuss each of this adjectives as we will explore the theory and use each of them later.

Our classical theory is based on decaying oscillator (oscillator that loses energy) Such oscillator has to follow Newton's second law.

Of course, our classical approach will not be able to explain all experimental results. More precise theory of dispersion will require quantum mechanical approach. At the same time, classical model is simple and allow to understand physical principle.

What forces are acting on electron?



„+" is too heavy => do not move

e/m wave only oscillate „-", i.e. electron.

1) Quasi-elastic force

$$\bar{F}_{Qe} = -k\bar{r}$$

It should remind you of Hooke's force

$\bar{r}$ - displacement

Why $\bar{r}$ is power 1? This is approximation and thus the system is considered linear.

Linear vs Nonlinear optics.

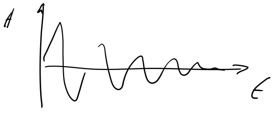
2) Radiation reaction force (Abraham-Lorentz force)

(analogue of viscosity friction).

$$F_{RF} = -\gamma\dot{r}$$

Why do we need to consider this force?

We have decaying oscillator, thus its amplitude decrease with time!



In classical mechanics we had to consider friction forces that are proportional to the body's speed.

Since we need to describe energy losses by oscillator, we formally consider radiation reaction force.

3) External driving force

$$\bar{F} = -e\bar{E}'$$

e - electron charge

Here $\bar{E}' = \bar{E} + \bar{E}_{in}$

$\bar{E}$ - external e/m wave.

$\bar{E}_{in}$ - internal field of other dipoles.

It is important here to consider low density medium => $\bar{E}_{in} = 0$

dipoles are too distant.

Thus, second Newton's law:

$$m\ddot{\bar{r}} = -k\bar{r} - \gamma\dot{\bar{r}} - e\bar{E}$$

$$\ddot{\bar{r}} + 2\beta\bar{r} + \omega_0^2\bar{r} = -\frac{e}{m}\bar{E}$$

$$\begin{cases} 2\beta = \frac{\gamma}{m} \\ \omega_0^2 = \frac{k}{m} \end{cases}$$

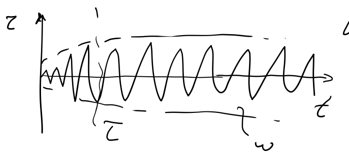
Since our field is harmonic wave:

$$\bar{E} = \bar{E}_0 e^{i\omega t}$$

There is no $k\bar{r}$, as atom is much smaller than λ, thus we can assume it is fully in the field of $\bar{E} = \bar{E}_0 e^{i\omega t}$ → dipole approximation

We will consider now equilibrated system

$t >> \tau$ τ is the time of equilibration



Once system is equillibrated, the oscillator will have frequency ω.

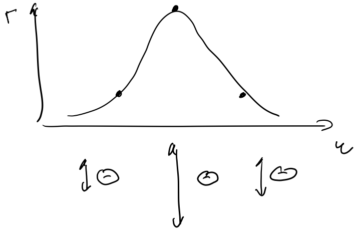
Second Newton's law will result in solution:

$$\bar{r}(\omega) = \frac{-\frac{e}{m}\bar{E}(\omega)}{}$$

$\omega_0^2-\omega^2+2i\beta\omega$

What does it mean?

The displacement $\bar{r}$ depends on how ω is different from ω_0. This is called resonance.



Atom now has the dipole moment $\bar{p}=-e\bar{r}$

$$\bar{p}(\omega)=\frac{e^2/m\ \bar{E}(\omega)}{\omega_0^2-\omega^2+2i\beta\omega}$$ – aslo function of ω.

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Let's N – concentration of the atoms.

$N=$ const $\Rightarrow$ homogeneous medium

If $N\neq$ const, then it will be turbid medium, i.e. medium, where refractive index at different spots is different $\Rightarrow$ scattering effects.

Medium polarization:

$$\bar{P}=N\bar{p}$$ $\rightarrow$ dipole moment of single atom

$P=\varepsilon_0\chi\bar{E}$ χ – dielectric susceptibility

We will only consider χ bieng scalar, i.e. $\bar{P}\parallel\bar{E}$. This means the medium is isotropic.

For anisotropic media χ is tensor.

Hence

$$\chi=\frac{Ne^2}{m\varepsilon_0}\,\frac{1}{\omega_0^2-\omega^2+2i\beta\omega}$$

In this case:

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P} = \varepsilon_0 \bar{E} + \varepsilon_0 \chi \bar{E} = \varepsilon_0 \underset{\overset{\parallel}{\varepsilon}}{(1+\chi)} \bar{E}$$

The dielectric constant:

$$\boxed{\varepsilon(\omega) = 1 + \chi = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}}$$

Let's summarize:

1) $\varepsilon = f(\omega)$

2) $\varepsilon(\omega)$ is complex function.

β is responsible for decay.

When there is no absorption => no energy losses => => no decay $\beta = 0$ => $\varepsilon(\omega)$ is real number.

For absorbing media:

$n' = \sqrt{\varepsilon} = n - i\varkappa$

n – refractive index

$\varkappa$ – decaying index

damping index

$$\varepsilon = n^2 - \varkappa^2 - 2in\varkappa$$

$$\begin{cases} n^2 - \varkappa^2 = 1 + \dfrac{Ne^2}{m\varepsilon_0} \dfrac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \longrightarrow \operatorname{Re}\varepsilon \\[2ex] n\varkappa = \dfrac{Ne^2}{m\varepsilon_0} \dfrac{\beta\omega}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \longrightarrow \operatorname{Im}\varepsilon \end{cases}$$

To simplify these equations we will need two assumptions:

1) When $\omega \simeq \omega_0$ (near the resonance):

$$\omega \simeq \omega_0$$

$$(\omega_0^2 - \omega^2)^2 = (\omega_0 + \omega)(\omega_0 - \omega) = 2\omega_0 \Delta\omega$$

2) We see, that n and $\varkappa$ are strongly intertwined. To decouple them, we can again take into consideration low density.

Taking into account first assumption:

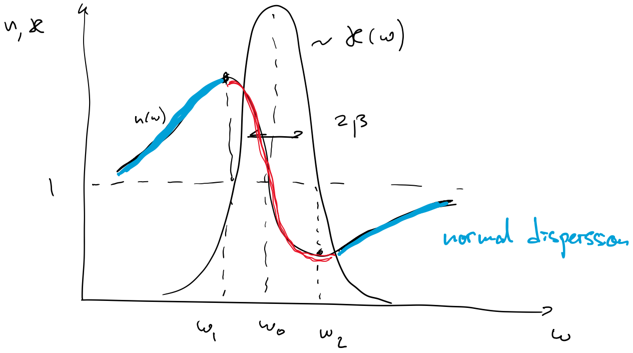
$$\begin{cases} n^2 - \varkappa^2 = 1 + \dfrac{Ne^2}{m\varepsilon_0}\cdot\dfrac{1}{2\omega_0}\cdot\dfrac{\Delta\omega}{\Delta\omega^2+\beta^2} \\ n\varkappa = \dfrac{Ne^2}{m\varepsilon_0}\cdot\dfrac{1}{4\omega_0}\cdot\dfrac{\beta}{(\Delta\omega)^2+\beta^2} \end{cases}$$

Since N is small $n\varkappa \ll 1$. Thus:

$n^2 - \varkappa^2 \approx 1$.

$$n \approx \sqrt{\varepsilon} \approx 1 + \frac{1}{2}\cdot\frac{Ne^2}{m\varepsilon_0}\cdot\frac{1}{2\omega_0}\cdot\frac{\Delta\omega}{\Delta\omega^2+\beta^2}$$

$$\begin{cases} n(\omega) \approx 1 + \dfrac{Ne^2}{m\varepsilon_0}\cdot\dfrac{1}{4\omega_0}\cdot\dfrac{\Delta\omega}{(\Delta\omega)^2+\beta^2} \\ \varkappa(\omega) \approx \dfrac{Ne^2}{m\varepsilon_0}\cdot\dfrac{1}{4\omega_0}\cdot\dfrac{\beta}{(\Delta\omega)^2+\beta^2} \end{cases}$$

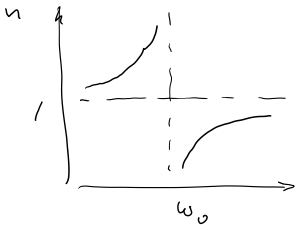


Most interesting to check refractive index away from ω_0.

In this case $|\omega_0^2 - \omega^2| \gg 2\beta\omega$

$$\boxed{\varepsilon = n^2 = 1 + \frac{Ne^2}{m\varepsilon_0}\,\frac{1}{\omega_0^2 - \omega^2}}$$

This is Zelmer formula



There is no physical meaning in discontinuation of $n(\omega)$. We simply do not consider when $n(\omega)$ is near resonance.